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THE RANGE OF THE GRADIENT OF A LIPSCHITZ C^1-SMOOTH BUMP IN INFINITE DIMENSIONS

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ABSTRACT

If a Banach space has a Lipschitz C^1-smooth bump function, then it admits other bumps of the same smoothness whose gradients exactly fill the dual unit ball and other reasonable figures. This strengthens a result of Azagra and Deville who were able to cover the dual unit ball.

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1. Introduction

A real-valued function on a Banach space with bounded nonempty support is called a **bump.** In [1], Azagra and Deville show that any Banach space with a Lipschitz C^1-smooth bump has another bump of the same smoothness whose set of gradients contains the dual unit ball. Here, applying subtler constructions, we use the same hypothesis to capture various closed sets in the dual—including the unit ball—as exact gradient images of suitable Lipschitz C^1-smooth bumps.

This note is an "infinite dimensional" continuation of [5], which concerns the range of the gradient of a C^1-smooth bump defined on $\mathbb{R}^n$. There are some significant differences between these two settings. In finite dimensions, the gradient range is automatically closed and contains the origin in its interior, whereas there exist Banach spaces with C^1-smooth norms (even reflexive ones) on which some bumps have gradient ranges with empty interior. (See [1] modulo [7], [2] and [3].) On the other hand, infinite dimensions allow us to position infinitely many bumps, with disjoint non-shrinking supports, inside a bounded set—a construction which is certainly impossible in $\mathbb{R}^n$.

TERMINOLOGY. In a Banach space X, we write $\mathbb{B}_X(x_0, r)$ for the closed ball with centre x_0 and radius r, and abbreviate $\mathbb{B}_X(0,1)$ as $\mathbb{B}_X$. The **support** of a bump $b\colon X \to \mathbb{R}$ is the set $\operatorname{supp} b = \{x \in X\colon b(x) \neq 0\}$. The **gradient** of a function f on X will mean its Fréchet derivative and be denoted by f' or ∇f. The range of a mapping F is denoted by $\mathcal{R}(F)$. Any sum of vectors indexed by the empty set is understood to equal the origin. Similarly, any product of numbers indexed by the empty set will be put equal to 1.

2. Tools

In this section we review five general strategies for manipulating C^1-smooth bumps, under headings A–E. Several of these have been used before, e.g., in [5].

A. SMOOTH COMPOSITION. In the Appendix, we construct C^∞-smooth functions p and m for use throughout this section. These analogues of the Lipschitz functions $t \mapsto (t \vee 0)$ and $(s,t) \mapsto (s \wedge t)$, respectively, allow us to mimic some standard operations without destroying C^1-smoothness.

We start by constructing a bump whose degree of smoothness is as good as that of the norm on the Banach space X in question. Clearly,

$$x \mapsto \big[4 - ((\|x\| - 2) \vee 0)\big] \vee 0, \quad x \in X,$$

is a Lipschitz bump that is as smooth as $\|\cdot\|$ at all points x obeying both $\|x\| \neq 2$ and $\|x\| \neq 6$. Replacing $(t \vee 0)$ with $p(t)$ yields the result below.

LEMMA 1: *Let X be a Banach space with a C^1-smooth norm $\|\cdot\|$. Then X admits a Lipschitz C^1-smooth bump b with $0 \le b(x) \le 4$ for every x, and $\mathcal{R}(\nabla b) = [0,1]\mathcal{R}(\|\cdot\|')$.*

Proof: Let p be provided by Lemma A with $r = 1$. We prove the stated conclusions for

$$b(x) = p\big(4 - p(\|x\| - 2)\big), \quad x \in X.$$

Note that p is nondecreasing and convex, with $p(\|x\| - 2) = 0$ if $\|x\| \le 2$, and $p(\|x\| - 2) = \|x\| - 3$ if $\|x\| \ge 4$, so

$$b(x) = \begin{cases} 3, & \text{if } \quad \|x\| \le 2, \\ 3 - p(\|x\| - 2), & \text{if } 2 < \|x\| < 4, \\ 6 - \|x\|, & \text{if } 4 \le \|x\| \le 5, \\ p(7 - \|x\|), & \text{if } 5 < \|x\| < 7, \\ 0, & \text{if } 7 \le \|x\|. \end{cases}$$

In particular, $\operatorname{supp} b \subseteq 7\mathbb{B}_X$, so b is a bump. The C^1-smoothness of b near 0 is evident; since p is C^∞-smooth, the chain rule [6, Chapter I, Theorem 5.4.2] immediately yields that b is C^1-smooth everywhere else.

Clearly $\mathcal{R}(\nabla b) \supseteq \mathcal{R}(\|\cdot\|')$. When $2 < \|x\| < 4$, the above expression for $b(x)$ gives

$$\nabla b(x) = -p'(\|x\| - 2)\,\|\cdot\|'(x).$$

There is a similar outcome when $5 < \|x\| < 7$. Since $p'([0,2]) = [0,1]$ and $\mathcal{R}(\|\cdot\|')$ is symmetric, we deduce that $\mathcal{R}(\nabla b) = [0,1] \cdot \mathcal{R}(\|\cdot\|')$. ∎

In any reflexive Banach space X with a C^1-smooth norm, we have $\mathcal{R}\left(\|\cdot\|'\right) = \partial\mathbb{B}_{X^*}$, and Lemma 1 provides a C^1-smooth bump b with $\mathcal{R}(\nabla b) = \mathbb{B}_{X^*}$. To guarantee this identity in C^1-smooth but nonreflexive spaces, or in reflexive spaces whose norm is not Fréchet smooth, more effort is required. This is the content of our main result, below; $\mathbb{B}_{X^*}$ is just one example of the gradient ranges we obtain.

B. TRUNCATION. Let $b\colon X \to \mathbb{R}$ be a bump with unbounded range, say, with $\sup_{x\in X} b(x) = +\infty$. Fix any $r > 0$, and apply Lemma A to produce a function p. Define

$$\widetilde{b}(x) = r - p\big(2r - p(b(x))\big), \quad x \in X.$$

We claim that $\widetilde{b}$ is a bump with range inside $[0,r]$. Indeed, take any $x \in X$. If $b(x) = 0$, then $p(b(x)) = 0$, so $\widetilde{b}(x) = r - p(2r) = 0$. Thus $\operatorname{supp}\widetilde{b} \subset \operatorname{supp} b$. If $p(b(x)) \ge 2r$, then $\widetilde{b}(x) = r - 0 = r$. If $(0 \le)\ p(b(x)) < 2r$, then $\widetilde{b}(x) \ge r - \frac{1}{2}(2r - p(b(x))) = \frac{1}{2}p(b(x)) \ge 0$. Hence $\mathcal{R}(\widetilde{b}) \subset [0,r]$. Now, take $x \in X$ such that $b(x) > 3r$. Then $\widetilde{b}(x) = r \ne 0$. Therefore $\widetilde{b}$ is a bump.

C. Domain-Scaling. For each $\beta > 0$, define an operator $T[\beta]$ on $C(X;\mathbb{R})$ as follows:

$$(T[\beta]\varphi)(x) = \beta\varphi\left(\beta^{-1}x\right), \quad x \in X.$$

If b is a C^1-smooth bump on X, with $\operatorname{supp} b \subset \mathbb{B}_X$, and $\beta > 0$ is given, then $T[\beta]b$ is a C^1-smooth bump on X with support in $\beta\mathbb{B}_X$, and satisfying $\mathcal{R}(\nabla T[\beta]b) = \mathcal{R}(\nabla b)$.

D. Domain-Shifting. Given two C^1-smooth bumps b_1, b_2 on X, a C^1-smooth bump whose gradient range equals $\mathcal{R}(\nabla b_1) \cup \mathcal{R}(\nabla b_2)$ is given by

$$x \mapsto b_1(x) + b_2(x-y), \quad x \in X,$$

where $y \in X$ is some fixed vector with sufficiently large norm.

E. Gradient-Range Chaining. Suppose b_1, b_2 are C^1-smooth bumps such that $\nabla b_1 \equiv \xi$ on a neighbourhood of some point $y \in X$. Then a C^1-smooth bump whose gradient range equals $\mathcal{R}(\nabla b_1) \cup \left(\xi + \mathcal{R}(\nabla b_2)\right)$ is given by fixing $\beta > 0$ sufficiently small and considering the function

$$x \mapsto b_1(x) + T[\beta]b_2(x-y), \quad x \in X.$$

In the following lemma, we construct C^1-smooth bumps to which this idea may be applied. Our notation is

$$D_\gamma(\alpha,\zeta) = \operatorname{co}\left(\mathbb{B}_{X^*}(\alpha,\gamma) \cup \{\alpha+\zeta\}\right)$$

for α, ζ in X^* and $\gamma > 0$. When $\|\zeta\| > \gamma$ (the case of interest), this is a "drop" whose body is centered at α and whose vertex is $\alpha + \zeta$.

Lemma 2 (Drop Lemma): *Let X be a Banach space with a Lipschitz C^1-smooth bump. Let $\zeta \in X^*$, $\gamma \in (0, \|\zeta\|)$, and $r > 0$ be given. Then there exist a constant $\beta > 0$ and a C^1-smooth bump $\varphi\colon X \to \mathbb{R}$ such that*

(i) *$0 \le \varphi(x) \le r$ for every $x \in X$,*
(ii) *$\varphi(x) = 0$ whenever $\|x\| \ge r$,*
(iii) *$\mathcal{R}(\nabla\varphi) \subseteq D_\gamma(0,\zeta)$, and*
(iv) *$\nabla\varphi(x) = \zeta$ for every $x \in \beta\mathbb{B}_X$.*

Proof: We treat the case $r = 1$. This suffices, for if some bump φ_1 satisfies (i)–(iv) for $r = 1$, and any other $r > 0$ is given, then $\varphi = T[r]\varphi_1$ satisfies (i)–(iv) as written.

By truncation, translation and scaling, as described above, we may assume that we have at hand a Lipschitz C^1-smooth bump b on X for which $b(0) > 0$,

$0 \le b(x) \le 1$, and $\|\nabla b(x)\| \le \gamma$ for every $x \in X$. Since $\|\nabla b(0)\| \le \gamma < \|\zeta\|$, there exists $x_1 \in X$ such that

$$-\langle \zeta, x_1 \rangle > \big(\frac{3}{4}\|\zeta\| + \frac{1}{4}\|\nabla b(0)\|\big)\|x_1\|.$$

Since $b(0) > 0$ and b is differentiable at 0, we may scale x_1 to arrange also that

$$\begin{aligned} &b(0) + \langle \zeta, x_1 \rangle > 0, \\ &b(x_1) - b(0) - \langle \nabla b(0), x_1 \rangle > -\frac{1}{4}\big(\|\zeta\| - \|\nabla b(0)\|\big)\|x_1\|. \end{aligned}$$

Now apply Lemma A with

$$r = \frac{1}{4}\big(\|\zeta\| - \|\nabla b(0)\|\big)\|x_1\|$$

to produce a function m for use in defining

$$g(x) = m\big(b(x), b(0) + \langle \zeta, x \rangle\big), \quad x \in X.$$

Clearly, g is a C^1-smooth function with $g(x) \le b(x) \le 1$ for every $x \in X$. We estimate

$$\begin{aligned} &b(x_1) - (b(0) + \langle \zeta, x_1 \rangle) \\ &\quad > \langle \nabla b(0), x_1 \rangle - \frac{1}{4}\big(\|\zeta\| - \|\nabla b(0)\|\big)\|x_1\| - \langle \zeta, x_1 \rangle \\ &\quad > \Big(-\|\nabla b(0)\| - \frac{1}{4}(\|\zeta\| - \|\nabla b(0)\|) + \frac{3}{4}\|\zeta\| + \frac{1}{4}\|\nabla b(0)\|\Big)\|x_1\| \\ &\quad = \frac{1}{2}\big(\|\zeta\| - \|\nabla b(0)\|\big)\|x_1\| = 2r. \end{aligned}$$

Continuity then provides some $\beta > 0$ such that

$$b(x) - \big(b(0) + \langle \zeta, x \rangle\big) > 2r \quad \text{for every } x \in x_1 + \beta \mathbb{B}_X,$$

and so, by the properties of m, $g(x) = b(0) + \langle \zeta, x \rangle$ and $\nabla g(x) = \zeta$ for these x. Since $\mathcal{R}(m') = \{(\alpha, 1-\alpha) \colon \alpha \in [0,1]\}$, the chain rule gives

$$\mathcal{R}(\nabla g) \subset \mathrm{co}[\mathcal{R}(\nabla b) \cup \{\zeta\}] \subset \mathrm{co}[\gamma \mathbb{B}_{X^*} \cup \{\zeta\}].$$

On diminishing β, if necessary, we may assume that $g(x) > \frac{1}{2}g(x_1)$ (> 0) whenever $x \in x_1 + \beta\mathbb{B}_X$. Let p be the function provided by Lemma A when $r = \frac{1}{4}g(x_1)$, and put

$$\varphi = p \circ g.$$

Then φ is C^1-smooth and $0 \le \varphi(x) \le 1$ for every $x \in X$, while the chain rule gives

$$\mathcal{R}(\nabla\varphi) \subset [0,1] \cdot \mathcal{R}(\nabla g) \subset \mathrm{co}[\gamma \mathbb{B}_{X^*} \cup \{\zeta\}] = D_\gamma(0,\zeta).$$

For every $x \in x_1 + \beta\mathbb{B}_X$, we have $g(x) > 2 \cdot \frac{1}{4} g(x_1)$, so $\varphi(x) = g(x) - \frac{1}{4} g(x_1)$, and $\nabla\varphi(x) = \nabla g(x) = \zeta$.

It remains to reposition the function φ and arrange its support and scaling properties. Observe that if $b(x) = 0$, then

$$(0 \le)\ \varphi(x) = p(g(x)) \le p(b(x) \wedge (b(0) + \langle z, x\rangle)) \le p(0)\ \ (= 0).$$

Hence $\operatorname{supp}\varphi \subseteq \operatorname{supp} b$, and the bump $x \mapsto (T[\gamma]\phi)(x - x_1)$ has all the desired properties for $\gamma > 0$ sufficiently small. (The β in conclusion (iv) equals γ times the β in the previous paragraph.) ∎

3. The main result

This section is devoted to the statement and proof of our main result, in which we construct a bump b whose gradient range is precisely $\overline{\Omega}$ for a given open connected set Ω in X^* satisfying some mild conditions. The central construction can be visualized by imagining a dense set of points in Ω, and realizing each point of $\overline{\Omega}$ as the limit of a countable chain of linked drops with vertices in the dense set. Each chain will be the gradient range of a countable pile of bumps constructed by the methods of Section 2. Building such a pile for each target point and then dispersing the piles throughout the unit ball of X produces the desired bump. Our argument was inspired by [1], [4] (and actually goes back to the proof of the Banach open mapping theorem), but includes safeguards against producing gradient images outside the set $\overline{\Omega}$.

THEOREM: *Let X be an infinite dimensional Banach space with a Lipschitz C^1-smooth bump. Let $\Omega \subset X^*$ be an open connected set containing the origin and satisfying this property:*

> *There exists a summable sequence $a_0, a_1, a_2, \ldots$ of positive numbers such that every $\eta \in \overline{\Omega}$ can be expressed as $\lim_{i\to\infty} \xi_i$ for some sequence $0 = \xi_0, \xi_1, \xi_2, \ldots$ in Ω such that $\|\xi_{i+1} - \xi_i\| < a_i$, and that the linear segment $\mathrm{co}\{\xi_i, \xi_{i+1}\}$ lies in Ω for every $i = 0, 1, 2, \ldots$.*

Then there exists a Lipschitz C^1-smooth bump $b\colon X \to [0,1]$ such that $\mathcal{R}(\nabla b) = \overline{\Omega}$.

Proof: Since X is C^1-smooth, the density character of X^* equals that of X. (This is a straightforward consequence of Ekeland's Variational Principle.) Thus, Ω must contain a dense subset, say D, whose cardinality equals the density character of X. Put $\Delta = \frac{1}{10}$, and apply [1, Lemma 2.1] to produce a subset $\{x_\xi : \xi \in D\}$ of $(1-\Delta)\mathbb{B}_X$ for which $\xi, \xi' \in D$ and $\xi \neq \xi'$ implies $\|x_\xi - x_{\xi'}\| \geq 4\Delta$.

Let $0 \neq \eta \in \overline{\Omega}$ be any fixed vector. The theorem assumption provides a sequence $0 = \xi_0, \xi_1, \xi_2, \ldots$ in Ω that converges to η and enjoys certain properties. It is an easy exercise to show that, by perturbing the entries slightly if needed, such a sequence can be chosen from elements of the set D. Let us assume that this has been done, and write $\eta(1) = \xi_1$, $\eta(2) = \xi_2, \ldots$. Thus each $\eta \in \overline{\Omega}$ identifies a sequence in D; for later convenience, we write $\eta(0) = 0$ in all cases.

It will be convenient to associate with every $\eta \in \overline{\Omega}$ the sequence of increments $(\zeta_1, \zeta_2, \ldots)$ defined by $\zeta_i = \eta(i) - \eta(i-1)$. (Note that $\|\zeta_{i+1}\| < a_i$ for every $i \geq 0$.) Every sequence $(\zeta_1, \zeta_2, \ldots)$ in X^* arising in this way from some $\eta \in \overline{\Omega}$ will be called an **admissible sequence**; an i-tuple $(\zeta_1, \ldots, \zeta_i)$ will be called **admissible** if it arises as the initial segment of some admissible sequence. We will write $\mathcal{A}$ for the collection of all admissible tuples, i.e.,

$$\mathcal{A} = \left\{ (\eta(1), \eta(2) - \eta(1), \eta(3) - \eta(2), \ldots, \eta(i) - \eta(i-1)) : \eta \in \overline{\Omega},\ i \in \mathbb{N} \right\} \cup \{\emptyset\}.$$

Finally, we define a set-valued mapping $S\colon \mathcal{A} \to 2^{X^*}$ as follows:

$$S(\zeta_1, \ldots, \zeta_i) = \{\zeta' \in X^* \colon (\zeta_1, \ldots, \zeta_i, \zeta') \in \mathcal{A}\}, \quad (\zeta_1, \ldots, \zeta_i) \in \mathcal{A}, \quad i \in \mathbb{N}.$$

(Consistency requires $S(\emptyset) = \{\zeta' \colon (\zeta') \in \mathcal{A}\} = \left\{\eta(1) \colon \eta \in \overline{\Omega}\right\}$.) Clearly $S(\cdot)$ is nonempty-valued on $\mathcal{A}$; we define $S(\zeta_1, \ldots, \zeta_i) = \emptyset$ for $(\zeta_1, \ldots, \zeta_i) \notin \mathcal{A}$.

The chains of drops mentioned in the preamble will have links of the form $D_\gamma(\alpha, \zeta)$, where $\alpha \in \Omega$ and $\zeta \in X^* \setminus \{0\}$ obey $\operatorname{co}\{\alpha, \alpha + \zeta\} \subset \Omega$. For every such pair, we fix $\gamma(\alpha, \zeta) > 0$ so small that $\gamma(\alpha, \zeta) < \|\zeta\|$ and $D_{\gamma(\alpha,\zeta)}(\alpha, \zeta) \subset \Omega$. Then we apply the Drop Lemma with $r = \Delta$, writing $\varphi_{\zeta,\gamma}$ for the resulting bump and $\beta_{\zeta,\gamma}$ for a number in $(0, \Delta)$ such that

$$\nabla \varphi_{\zeta,\gamma}(z) = \zeta \quad \text{whenever } z \in \beta_{\zeta,\gamma} B_X. \tag{1}$$

We shall use the following simplified notation. For any admissible sequence $(\zeta_1, \zeta_2, \ldots)$ associated with a point $\eta \in \overline{\Omega}$, and any $i \in \mathbb{N}$, we put

$$\begin{aligned} \gamma_i &= 2^{1-i}\gamma(\zeta_1 + \cdots + \zeta_{i-1}, \zeta_i), \\ \beta_i &= \beta_{\zeta_i,\gamma_i}, \\ \widehat{x}_i &= x_{\eta(1)} + \beta_1 x_{\eta(2)} + \beta_1\beta_2 x_{\eta(3)} + \cdots + \beta_1 \cdots \beta_{i-1} x_{\eta(i)}. \end{aligned} \tag{2}$$

(In particular, we put $\gamma_1 = \gamma(0,\zeta_1)$.) Notice that whenever $\eta,\eta' \in \overline{\Omega}$ generate admissible sequences whose first i entries coincide, these definitions are consistent. Thus, in any formula containing an admissible i-tuple $(\zeta_1,\ldots,\zeta_i)$ together with such symbols as $\gamma_1,\ldots,\gamma_i$, $\beta_1,\ldots,\beta_i$, or $\widehat{x}_1,\ldots,\widehat{x}_i$, we understand that the relationships in (2) are in force.

Along any admissible sequence arising from some $\eta \in \overline{\Omega}$, the inequalities $\|x_{\eta(i)}\| < 1-\Delta$ and $\beta_i < \Delta$ hold for every $i \in \mathbb{N}$, so (2) gives

$$\|\widehat{x}_i\| < (1-\Delta)(1+\Delta+\cdots+\Delta^{i-1}) = 1-\Delta^i. \tag{3}$$

The sequence $(\widehat{x}_1,\widehat{x}_2,\ldots)$ provides a list of centres for a nested family of balls in X:

$$\begin{aligned}\operatorname{int}\mathbb{B}(\widehat{x}_i,\beta_1\cdots\beta_{i-1}\Delta) &\supseteq \operatorname{int}\mathbb{B}(\widehat{x}_i,\beta_1\cdots\beta_{i-1}\beta_i)\\ &\supseteq \mathbb{B}(\widehat{x}_{i+1},\beta_1\cdots\beta_i\Delta), \quad i=0,1,2,\ldots.\end{aligned} \tag{4}$$

Indeed, the first inclusion holds because $\beta_i < \Delta$. For the second, note that each x in $\mathbb{B}(\widehat{x}_{i+1},\beta_1\cdots\beta_i\Delta)$ obeys

$$\begin{aligned}\|x-\widehat{x}_i\| &\le \|x-\widehat{x}_{i+1}\| + \beta_1\cdots\beta_i\,\|x_{\eta(i+1)}\|\\ &< \beta_1\cdots\beta_i\Delta + \beta_1\cdots\beta_i(1-\Delta) = \beta_1\beta_2\cdots\beta_i.\end{aligned}$$

Each nested sequence of balls described above supports one of the "countable piles of bumps" mentioned in the preamble.

Now suppose **distinct** η,η' in $\overline{\Omega}$ are given, with associated scaling sequences (β_i), (β_i') and centres $(\widehat{x}_i)$, $(\widehat{x}_i')$, respectively. Choose the largest $p \in \mathbb{N}$ such that $\eta(j) = \eta'(j)$ for $j \le p$. Clearly $\beta_j' = \beta_j$ and $\widehat{x}_j' = \widehat{x}_j$ for $j \le p$; then, at stage $p+1$, the nested sequences of balls corresponding to η and η' split apart as follows:

$$\left.\begin{aligned} y &\in \mathbb{B}(\widehat{x}_{p+1},\beta_1\cdots\beta_p\Delta)\\ y' &\in \mathbb{B}(\widehat{x}_{p+1}',\beta_1\cdots\beta_p\Delta)\end{aligned}\right\} \implies \|y-y'\| > \beta_1\cdots\beta_p\Delta. \tag{5}$$

Indeed, since distinct points of the form x_ξ, $\xi \in D$, are separated by at least 4Δ, we have

$$\left\|\widehat{x}_{p+1}-\widehat{x}_{p+1}'\right\| = \beta_1\cdots\beta_p\left\|x_{\eta(p+1)} - x_{\eta'(p+1)}\right\| \ge 4\beta_1\cdots\beta_p\Delta,$$

so any y, y' as in (5) obey

$$\|y'-y\| \ge \left\|\widehat{x}_{p+1}-\widehat{x}_{p+1}'\right\| - \|y-\widehat{x}_{p+1}\| - \left\|y'-\widehat{x}_{p+1}'\right\| \ge \beta_1\cdots\beta_p(4\Delta-\Delta-\Delta).$$

This proves (5). In conjunction with (4), it implies that whenever $i,j \ge p+1$,

$$\left.\begin{aligned} y &\in \mathbb{B}(\widehat{x}_i,\beta_1\cdots\beta_{i-1}\Delta)\\ y' &\in \mathbb{B}(\widehat{x}_j',\beta_1'\cdots\beta_{j-1}'\Delta)\end{aligned}\right\} \implies \|y-y'\| > \beta_1\cdots\beta_p\Delta \ge \beta_1\cdots\beta_{i-1}\Delta. \tag{6}$$

Now for each $n \in \mathbb{N}$ we define a bump $b_n \colon X \to \mathbb{R}$ by nesting n sums:

$$b_n(x) = \sum_{\zeta_1 \in S(\emptyset)} \Bigg(\varphi_{\zeta_1,\gamma_1}(x - \widehat{x}_1) + \sum_{\zeta_2 \in S(\zeta_1)} \Bigg(T[\beta_1]\varphi_{\zeta_2,\gamma_2}(x - \widehat{x}_2) \ddots + \sum_{\zeta_n \in S(\zeta_1,\zeta_2,\dots,\zeta_{n-1})} T[\beta_1\beta_2\cdots\beta_{n-1}]\varphi_{\zeta_n,\gamma_n}(x - \widehat{x}_n) \Bigg) \cdots \Bigg). \tag{7}$$

To clarify the support of b_n, recall that for every $\zeta \in X^*$ and $\gamma \in (0, \|\zeta\|)$, we have $\varphi_{\zeta,\gamma}(z) = 0$ whenever $\|z\| \geq \Delta$. Hence for each $i \leq n$,

$$T[\beta_1\beta_2\cdots\beta_{i-1}]\varphi_{\zeta_i,\gamma_i}(x - \widehat{x}_i) \neq 0 \quad \Longrightarrow \quad \|x - \widehat{x}_i\| < \beta_1\cdots\beta_{i-1}\Delta < \Delta^i. \tag{8}$$

Every $x \in X$ with $\|x\| \geq 1$ satisfies $\|x - \widehat{x}_i\| > \Delta^i$ by (3), and hence produces a zero value in every term on the right side of (7). Thus $\operatorname{supp} b_n \subseteq \mathbb{B}_X$. Also, in view of (1), we have

$$\nabla\left(T[\beta_1\beta_2\cdots\beta_{i-1}]\varphi_{\zeta_i,\gamma_i}\right)(x - \widehat{x}_i) = \zeta_i \quad \text{whenever } \|x - \widehat{x}_i\| < \beta_1\cdots\beta_{i-1}\beta_i. \tag{9}$$

We have taken care to arrange the following situation:

Claim: *For every $x \in X$ and every $n \in \mathbb{N}$ there are an admissible n-tuple $(\zeta_1, \dots, \zeta_n)$ and a neighbourhood U of x such that all $y \in U$ satisfy*

$$\begin{aligned} b_n(y) = {} & \varphi_{\zeta_1,\gamma_1}(y - \widehat{x}_1) \\ & + T[\beta_1]\varphi_{\zeta_2,\gamma_2}(y - \widehat{x}_2) \\ & + T[\beta_1\beta_2]\varphi_{\zeta_3,\gamma_3}(y - \widehat{x}_3) \\ & + \cdots + T[\beta_1\beta_2\cdots\beta_{n-1}]\varphi_{\zeta_n,\gamma_n}(y - \widehat{x}_n), \end{aligned} \tag{10}$$

and, in particular,

$$\nabla b_n(x) \in \Omega. \tag{11}$$

To prove this claim, fix $x \in X$ and $n \in \mathbb{N}$. Suppose first that x lies outside the ball $\mathbb{B}(x_{\eta(1)}, \Delta)$ for every $\eta \in \overline{\Omega}$. Since $\|x_\xi - x_{\xi'}\| \geq 4\Delta$ whenever $\xi \neq \xi'$ in D, the triangle inequality implies that $U_0 = \{y \colon \|y - x\| < \Delta\}$ meets at most one ball of the form $\mathbb{B}(x_\xi, \Delta)$, $\xi \in D$. Hence there exists $\delta > 0$ for which $U = \mathbb{B}(x, \delta)$ meets no such ball. Combining (8) and (4) establishes that $b_n(y) = 0$ for every $y \in U$, and moreover, that in this degenerate case, (10) holds for **every** admissible n-tuple. Of course, $\nabla b_n(x) = 0$ obeys (11).

Alternatively, suppose $x \in \mathbb{B}(x_{\eta(1)}, \Delta)$ for some $\eta \in \overline{\Omega}$. This establishes case $p = 1$ of the following condition:

$$x \in \mathbb{B}(\widehat{x}_p, \beta_1 \cdots \beta_{p-1}\Delta) \quad \text{for some } (\zeta_1, \ldots, \zeta_p) \in \mathcal{A}, \quad 1 \le p \le n.$$

Choose the largest such p. Condition (6) implies that the associated p-tuple $(\zeta_1, \ldots, \zeta_p)$ is unique. Moreover, using this p-tuple to specify

$$U_0 = \{y\colon \|y - x\| < \beta_1 \cdots \beta_p \Delta\}$$

produces an open set in which the only nonzero summands in lines 1 through p of (7) are associated with this same p-tuple. If $p = n$, this establishes (10). If $p < n$, the maximality of p implies that for every $\zeta' \in S(\zeta_1, \ldots, \zeta_p)$, we have $x \notin \mathbb{B}(\widehat{x}'_{p+1}, \beta_1 \cdots \beta_p \Delta)$. Recall that, by (2), $\widehat{x}'_{p+1} = \widehat{x}_p + \beta_1 \cdots \beta_p x_{\eta'(p+1)}$. The 4Δ-separation of distinct vectors of form x_ξ, $\xi \in D$, guarantees that among all the balls $\mathbb{B}(\widehat{x}'_{p+1}, \beta_1 \cdots \beta_p \Delta)$ arising this way, there can be at most one that meets U_0. And since the centre of U_0 lies outside that one, we can choose $\delta > 0$ so small that $U = \{y\colon \|y - x\| < \delta\}$ obeys

$$U \cap \mathbb{B}(\widehat{x}'_{p+1}, \beta_1 \cdots \beta_p \Delta) = \emptyset \quad \text{whenever } \zeta' \in S(\zeta_1, \ldots, \zeta_p).$$

Together with (6) and (8), this shows that all summands in lines $p+1, \ldots, n$ of (7) contribute 0 to the definition of $b_n(y)$, for every y in U. Indeed, much as in the previous paragraph, conclusion (10) holds for U along any admissible n-tuple whose first p entries agree with the given ones. To prove (11), recall (4): for each $j < p$ we have $\|x - \widehat{x}_j\| < \beta_1 \cdots \beta_j$, so by (9),

$$(12) \qquad \nabla b_n(x) \in \zeta_1 + \zeta_2 + \cdots + \zeta_{p-1} + D_{\gamma_p}(0, \zeta_p) = D_{\gamma_p}(\eta(p-1), \eta(p)) \subseteq \Omega.$$

Thus the stated claim holds in all cases.

The claim above implies that each b_n is C^1-smooth on X; also, for all $x \in X$,

$$|b_{n+1}(x) - b_n(x)| < \Delta^n \quad \textit{and} \quad \|\nabla b_{n+1} - \nabla b_n(x)\| < 2^{-n} + a_n.$$

Thus the bumps b_n, $n \in \mathbb{N}$, converge uniformly to a bump, say b, and their gradients ∇b_n, $n \in \mathbb{N}$, also converge uniformly on X. Therefore b is a C^1-smooth function with $\nabla b(x) = \lim_{n\to\infty} \nabla b_n(x)$ uniformly for $x \in X$. We note that $\operatorname{supp} b \subset \mathbb{B}_X$ and that $0 \le b(x) < \Delta + \Delta^2 + \cdots < 1$ as $\Delta = \frac{1}{10}$.

Since $\mathcal{R}(\nabla b_n) \subseteq \Omega$ for every n by (11), it follows that $\mathcal{R}(\nabla b) \subseteq \overline{\Omega}$.

To show that $\mathcal{R}(\nabla b) \supseteq \overline{\Omega}$, fix any $\eta \in \overline{\Omega}$. Let $(\zeta_1, \zeta_2, \ldots)$ be the corresponding admissible sequence, with γ_i, scale factors β_i and centres $\widehat{x}_i$ given by (2). Note that since each $\beta_i < \Delta = \frac{1}{10}$ and $\|x_\xi\| < 1 - \Delta$, the centres $\widehat{x}_i$ are partial sums of the convergent series

$$\overline{x} := x_{\eta(1)} + \beta_1 x_{\eta(2)} + \beta_1\beta_2 x_{\eta(3)} + \cdots.$$

Moreover,

$$\begin{aligned}
\|\overline{x} - \widehat{x}_1\| \leq &(\beta_1 + \beta_1\beta_2 + \cdots)(1-\Delta) \\
&< \beta_1(1 + \Delta + \Delta^2 + \cdots)(1-\Delta) = \beta_1, \\
\|\overline{x} - \widehat{x}_2\| \leq &(\beta_1\beta_2 + \beta_1\beta_2\beta_3 + \cdots)(1-\Delta) \\
&< \beta_1\beta_2(1 + \Delta + \Delta^2 + \cdots)(1-\Delta) = \beta_1\beta_2, \\
&\vdots
\end{aligned}$$

so $\overline{x} \in \mathbb{B}(\widehat{x}_i, \beta_1 \cdots \beta_i)$ for every $i \in \mathbb{N}$. This implies that $\nabla b_n(\overline{x}) = \zeta_1 + \cdots + \zeta_n$ for every $n \in \mathbb{N}$, and

$$\nabla b(\overline{x}) = \lim_{n\to\infty} \nabla b_n(\overline{x}) = \lim_{n\to\infty} \big(\zeta_1 + \cdots + \zeta_n\big) = \lim_{n\to\infty} \eta(n) = \eta.$$

Since $\eta \in \overline{\Omega}$ was arbitrary, we have $\overline{\Omega} \subseteq \mathcal{R}(\nabla b)$. ∎

Remarks: Our Theorem evidently applies whenever $\Omega \subset X^*$ is an open bounded convex set containing the origin.

Consider any set $U \subset X^*$ that can be expressed as $\bigcup_{\alpha\in A} \overline{\Omega_\alpha}$ where each Ω_α satisfies the assumptions of the Theorem above, and where A is a set whose cardinality does not exceed the density character of X. For every $\alpha \in A$ let b_α be a C^1-smooth bump on X such that $\mathcal{R}(\nabla b_\alpha) = \overline{\Omega_\alpha}$. By scaling and shifting domains, we may arrange for all b_α's to have mutually disjoint supports lying in B_X. Then $b = \sum_{\alpha\in A} b_\alpha$ is a C^1-smooth (but perhaps not Lipschitz) bump satisfying $\mathcal{R}(\nabla b) = U$. This construction covers many unbounded sets, even in the case where the index set A is countable. In particular, we can express any open connected set U containing the origin as $\mathcal{R}(\nabla b)$, as Azagra and Jimenez have shown that all such sets have the above form. Taking $U = X^*$ reproduces the result of Azagra and Deville [1].

Our theorem allows for sets Ω that are far from being starshaped, like the interior of $\mathbb{B}_X(x_0, 2) \setminus \mathbb{B}_X(x_0, 1)$ whenever $1 < \|x_0\| < 2$. Indeed, gradient-range chaining (tool E above) allows us to produce figures which are not even simply connected. A finite-dimensional version of this observation, with illustrations, appears in [5].

The conclusions of our Theorem cover those of [5, Theorem 12], one of that paper's main results. However, the proof in [5] (where $X = \mathbb{R}^n$) needs a rather different argument because no bounded subset of $\mathbb{R}^n$ can support infinitely many bumps having disjoint supports of comparable diameters.

4. Appendix

Here we construct C^∞-smooth functions p and m that imitate the functions $t \mapsto t^+$ and $(s,t) \mapsto s \wedge t$.

LEMMA A: *Let $r > 0$, and let $\rho : \mathbb{R} \to [0, +\infty)$ be an even C^∞-smooth function, with $\operatorname{supp} \rho \subset [-r, r]$ and $\int_{\mathbb{R}} \rho(u)\mathrm{d}u = 1$. Define $p: \mathbb{R} \to [0, +\infty)$ and $m: \mathbb{R}^2 \to \mathbb{R}$ by*

$$p(t) = \int_{\mathbb{R}} [t - r - u]^+ \rho(u)\mathrm{d}u, \quad t \in \mathbb{R},$$
$$m(s,t) = \iint_{\mathbb{R}^2} [(s-u) \wedge (t-v)]\, \rho(u)\rho(v)\mathrm{d}u\mathrm{d}v, \quad (s,t) \in \mathbb{R}^2.$$

Then p is nondecreasing and convex, both p and m are C^∞-smooth, and

$$p(t) = \begin{cases} 0, & \text{if } t \le 0, \\ t - r, & \text{if } t \ge 2r, \end{cases} \qquad 0 \le p(t) \le \frac{1}{2} t \text{ if } 0 < t \le 2r,$$
$$m(s,t) = m(t,s) = \begin{cases} s, & \text{if } s \le t - 2r, \\ t, & \text{if } t \le s - 2r, \end{cases} \qquad m(s,t) \le s \wedge t \text{ always},$$

and $\mathcal{R}(p') = [0,1]$ and $\mathcal{R}(m') = \{(\alpha, 1-\alpha) : \alpha \in [0,1]\}$.

Proof: The smoothness of p and m is well-known and easy to prove. If $t \ge 2r$, we have $t - r - u \ge 0$ for every $u \in \operatorname{supp} \rho$ and hence

$$p(t) = \int_{\mathbb{R}} (t - r - u)\rho(u)\mathrm{d}u = t - r - \int_{\mathbb{R}} u\rho(u)\mathrm{d}u = t - r,$$

because $u \mapsto u\rho(u)$ is an odd function. Similarly, $p(t) = 0$ for $t \le 0$. Since $t \mapsto (t-r)^+$ is convex, p is convex too: we have $p(0) = 0$ and $p(2r) = r$, so this forces $p(t) \le \frac{1}{2}t$ for $0 \le t \le 2r$.

For any $t \in [0, 2r]$, the following calculation shows that $p'(t) \in [0,1]$:

$$p'(t) = \int_{\mathbb{R}} [t - r - u]^+ \rho'(u)\mathrm{d}u = \int_{-r}^{t-r} (t - r - u)\rho'(u)\mathrm{d}u = \int_{-r}^{t-r} \rho(u)\mathrm{d}u.$$

Since $p \in C^\infty$, with $p'(t) = 0$ for $t < 0$ and $p'(t) = 1$ for $t > r$, we have $\mathcal{R}(p') = [0, 1]$.

Similar arguments establish the stated properties of m. Indeed, the gradient of $(s,t) \mapsto s \wedge t$ lies in $\{(0,1),(1,0)\}$ whenever it exists, so the averaging implicit in mollification gives $\nabla m(s,t) \in \operatorname{co}\{(0,1),(1,0)\} = \{(\alpha, 1-\alpha) \colon \alpha \in [0,1]\}$ for every (s,t). But ∇m is continuous and takes on the values $(1,0)$ and $(0,1)$, so $\mathcal{R}(\nabla m) = \operatorname{co}\{(0,1),(1,0)\}$. ∎

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